Mean-Field Ferromagnet [tln84]

Macroscopic specification:

– caloric equation of state: $C_M = 0$,

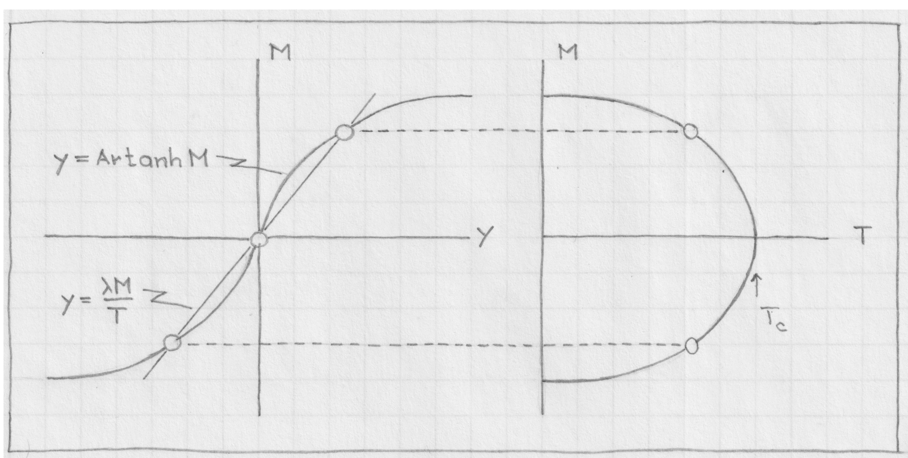
– thermodynamic equation of state: $M = \tanh\left(\dfrac{H + \lambda M}{T}\right)$.

The mean-field term, λM, represents a positive feedback, mimicking a spin-spin interaction favoring alignment.

The Langevin paramagnet ($\lambda = 0$) is solved in [tex21] from a thermodynamic vantage point and in [tex85] as a statistical mechanics problem.

Magnetization: $M(T, H)$

It is instructive to examine the graphical solution of the transcendental equation, $\mathrm{Artanh}\, M = \lambda M/T$, as sketched below.



– $M = 0$: solution with full symmetry at all T,

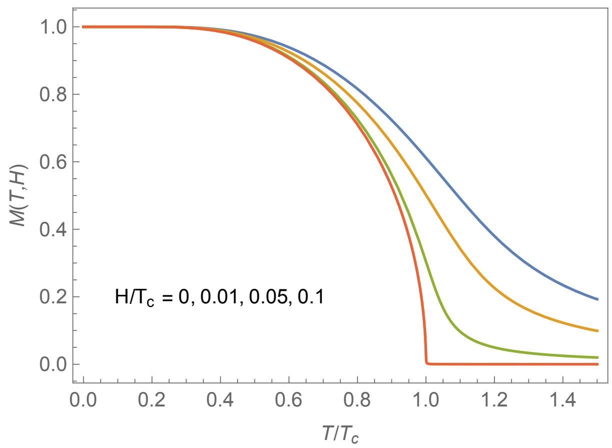
– $M \neq 0$: solutions with broken symmetry at $T < T_\mathrm{c} = \lambda$.

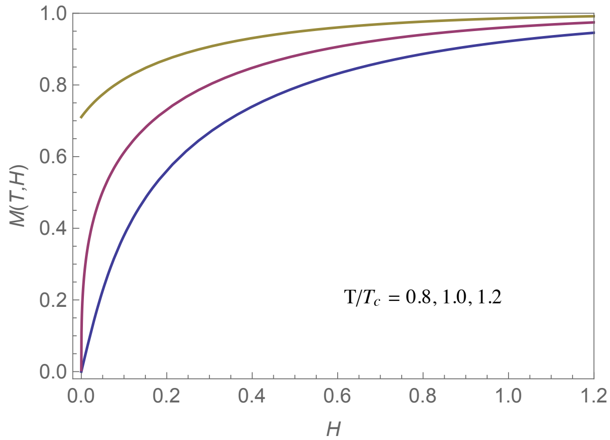
Critical singularity (square-root cusp) identified via expansion:

$$\mathrm{Artanh}\, M = M + \frac{1}{3}M^3 + \cdots = \lambda M/T \quad \Rightarrow \; M \sim \sqrt{3\left(\frac{T_\mathrm{c}}{T} - 1\right)}.$$

The solution representing thermal equilibrium, $M = 0$ at $T \geq T_\mathrm{c}$ and $M \neq 0$ at $T < T_\mathrm{c}$, minimizes the Helmholtz free energy $A(T, M)$ (analyzed below).

A numerical solution of $M(T,H)$ is worked out in [tex45], yielding the curves shown in the plots below.





– The magnetization at $T < T_c$ is only truly spontaneous for $H = 0$. It is merely enhanced for $H \neq 0$.

– M versus H at constant T are named magnetization curves.

– Magnetization curves change their shape qualitatively:

$T > T_c$: no intercept, finite initial slope,

$T = T_c$: no intercept, infinite initial slope,

$T < T_c$: intercept, finite initial slope,

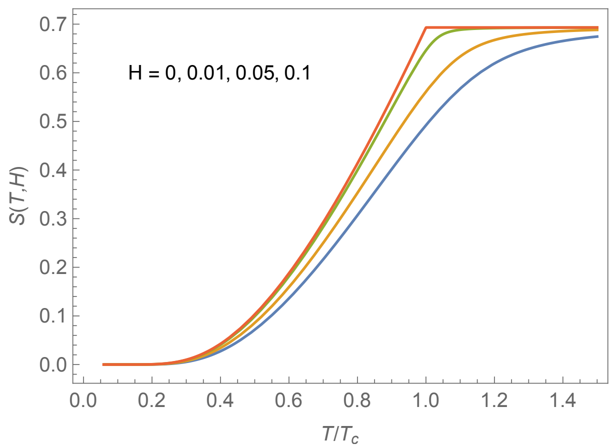
– Cusp singularity of the critical magnetization curve:

$$H = \text{Artanh}M - M = \frac{1}{3}M^3 + \ldots \quad \Rightarrow \quad M \sim (3H)^{1/3}.$$

Entropy: $S(T, H)$

The dependence on T and H of the entropy is entirely encoded in the function $M(T, H)$ analyzed earlier and in the function $S(M)$ derived in [tex45]:

$$S(M) = -\frac{1+M}{2}\ln\frac{1+M}{2} - \frac{1-M}{2}\ln\frac{1-M}{2}.$$



– The low-temperature limit is consistent with the third law:

$$\lim_{T\to 0} S(T, H) = 0.$$

– The T-dependence of $S(T, H)$ at $H \neq 0$ is smooth and monotonic.

– The function $S(T, 0)$ has a linear-cusp singularity at T_c [tex45].

– The constant function $S(T, 0)$ at $T > T_c$ signals the inability of the system to absorb further heat.

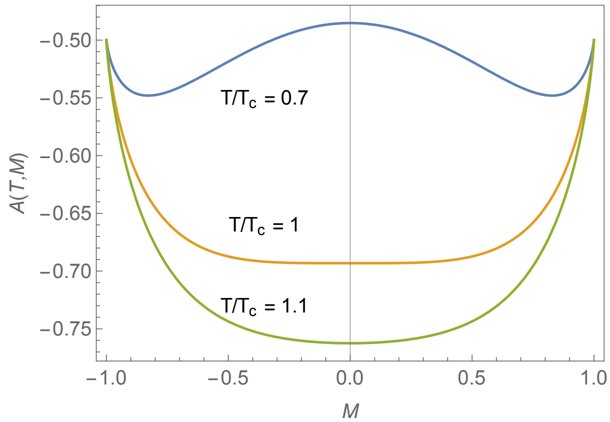
Helmholtz free energy: $A(T, M)$

The Helmholtz free energy is determined via integration of its differential,

$$dA = -SdT + HdM,$$

along a specific path in the (T, M)-plane, by use of the known functions $S(M)$ and $H(T, M)$ [tex45]:

$$A(T, M) = T\left[\frac{1+M}{2}\ln\frac{1+M}{2} + \frac{1-M}{2}\ln\frac{1-M}{2}\right] - \frac{1}{2}T_cM^2.$$



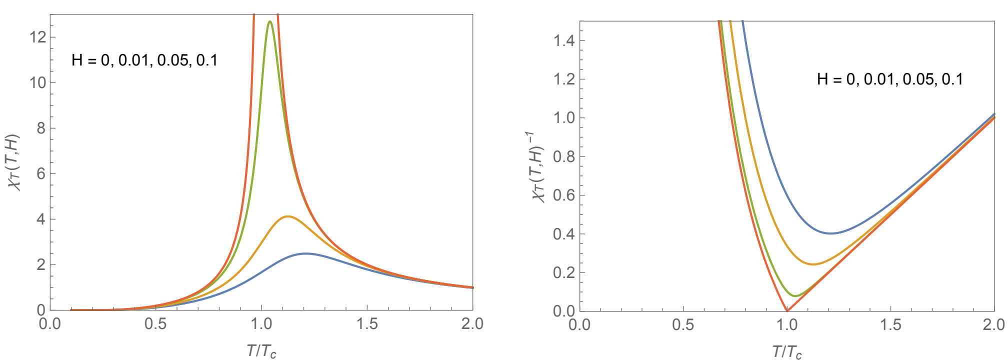
– The entropy $S(M)$ and the thermodynamic equation of state $H(T, M)$ are readily recovered from first partial derivatives.

– In the absence of a magnetic field, $A(T, M)$ at given T will assume its minimum value: at $M = 0$ for $T \geq T_c$ and at $M \neq 0$ for $T < T_c$.

Isothermal susceptibility: $\chi_T(T, H) \doteq \left(\frac{\partial M}{\partial H}\right)_T$

Magnetic response function of the mean-field ferromagnet from [tex46]:

$$\chi(T, H) = \left[\frac{T}{1 - M^2} - T_c\right]^{-1}, \quad M = M(T, H).$$



– The system is highly susceptible in the vicinity of T_c.

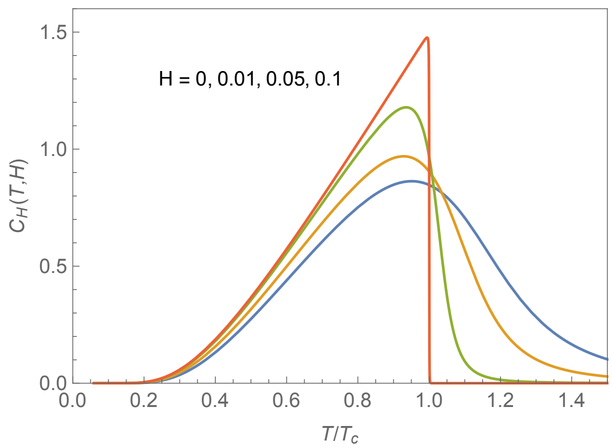
– The susceptibility is suppressed at high T, where the influence of the magnetic field is weak, and at low T where the ordering is stable.

– The susceptibility has a strong power-law divergence at T_c.

Heat capacity: $C_H(T,H) \doteq T\left(\frac{\partial S}{\partial T}\right)_H$

Thermal response function of the mean-field ferromagnet from [tex46]:

$$C_H = \left[\frac{1}{1-M^2} - \frac{T_c}{T}\right]^{-1} [\mathrm{Artanh} M]^2, \quad M = M(T,H).$$



- Unlike the identically vanishing C_M, the heat capacity C_H has a non-trivial profile.
- C_H vanishes in the low-T limit in accordance with the third law.
- At $H \neq 0$ there is capacity for absorbing heat at all $T > 0$.
- At $H = 0$ there is only heat capacity at $T \leq T_c$.
- The singularity at T_c of this response function is a discontinuity.