Equivalent-neighbor Ising model [tln98]

Consider an array of N localized spins $\sigma_i = \pm 1$ with an equivalent Ising-like interaction between all pairs (Husimi-Temperley model).

$$\mathcal{H} = -\frac{J}{N}\sum_{i<j}\sigma_i\sigma_j = -\frac{J}{2N}\left\{\sum_{i=1}^{N}\sum_{j=1}^{N}\sigma_i\sigma_j + \sum_{i=1}^{N}\sigma_i^2\right\} = \frac{J}{2}\left\{1 - \frac{1}{N}\left[\sum_{i=1}^{N}\sigma_i\right]^2\right\}.$$

The spatial arrangement of the array including its dimensionality is arbitrary. A meaningfult thermodynamic limit requires that the coupling strength is inversely proportional to N.

Canonical partition function:

$$Z_N = e^{-K/2}\sum_{\{\sigma_i\}}\exp\left(\frac{K}{2N}\left[\sum_{i=1}^{N}\sigma_i\right]^2\right), \quad K = \frac{J}{k_BT}.$$

Mathematical identity:

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}dx e^{-(x-a)^2/2} = 1 \quad \Rightarrow\ e^{(a^2)} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}dx\exp\left(-\frac{1}{2}x^2 + \sqrt{2}ax\right).$$

$$\Rightarrow\ Z_N = \frac{e^{-K/2}}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}dx\, e^{-x^2/2}\underbrace{\sum_{\{\sigma_i\}}\exp\left(\sqrt{\frac{K}{N}}x\sum_{i=1}^{N}\sigma_i\right)}_{\left[2\cosh\left(\sqrt{\frac{K}{N}}x\right)\right]^N}.$$

Variable transformation: $x = y\sqrt{KN}$:

$$\Rightarrow\ Z_N = e^{-K/2}\sqrt{\frac{KN}{2\pi}}\int_{-\infty}^{+\infty}dy\, e^{-KNy^2/2}\left[2\cosh\left(Ky\right)\right]^N$$
$$= 2^N e^{-K/2}\sqrt{\frac{KN}{2\pi}}I(K,N),$$

$$I(K,N) = \int_{-\infty}^{+\infty}dy\, e^{Nf(K,y)}, \quad f(K,y) = -\frac{1}{2}Ky^2 + \ln\big(\cosh(Ky)\big).$$

This integral can be evaluated asymptotically for large N by the Laplace method (saddle-point integral).

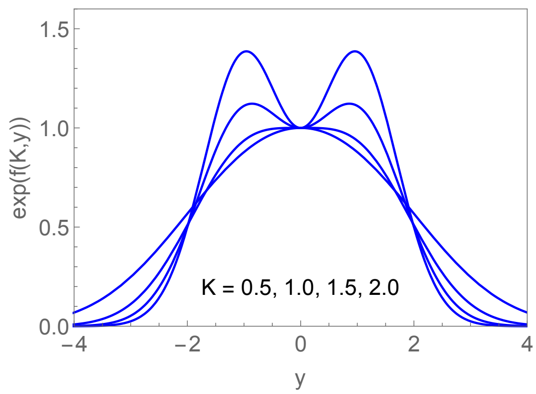
$$I(K,N) \rightsquigarrow A_N e^{NF(K)}, \quad F(K) = \max f(K,y), \quad \lim_{N\to\infty}N^{-1}\ln A_N = 0.$$

Gibbs free energy per site of the array (with no magnetic field):

$$\bar{G}(T, H=0) = -k_BT\lim_{N\to\infty}\left[N^{-1}\ln Z_N\right] = -k_BT\big[\ln 2 + F(K)\big].$$

Extrema of function $f(K,y) = -\frac{1}{2}Ky^2 + \ln\big(\cosh(Ky)\big)$.

- $K \leq 1$: one maximum at $y = 0$,
- $K > 1$: two maxima at $y = \pm y_0$.



The value y_0 (order parameter) is the solution of $y_0 = \tanh(Ky_0)$.

The dependence of y_0 on temperature is functionally equivalent to the mean-field solution of the Ising model: $\bar{M} = \tanh(\beta zJ\bar{M})$ with $zJ = k_B T_{\rm MF}$.

Thermal fluctuations are more efficiently suppressed by interactions of longer range than shorter range. Mean-field results are known to be more accurate away from strong thermal fluctuations.

The equivalent-neighbor Ising model can be interpreted as a model of infinite-range interactions. implying very strong suppression of thermal fluctuations. Unsurprisingly then, the spontaneous ordering is mean-field like.