[tex109] Fluctuations in a magnetic system

Consider a quantum magnet. The Hamiltonian is of the form $\mathcal{H} = \mathcal{H}_{\text{int}} - HM$, where $\mathcal{H}_{\text{int}}$ describes the (unspecified) interaction between microscopic magnetic moments, H is the magnitude of the external magnetic field (assumed constant) and M is the component of the total magnetic moment in the direction of the field. Given the Gibbs free energy $G(T, H, N) = -k_B T \ln Z_N$ as derived from the canonical partition function $Z_N = \operatorname{Tr} e^{-\beta\mathcal{H}}$, where $\beta = (k_B T)^{-1}$, derive the following relations (a) between energy fluctuations and heat capacity at constant field,

$$\langle \mathcal{H}^2 \rangle - \langle \mathcal{H} \rangle^2 = \frac{\partial^2}{\partial \beta^2} \ln Z_N = k_B T^2 C_h,$$

and (b) between magnetisation fluctuations and isothermal susceptibility,

$$\langle M^2 \rangle - \langle M \rangle^2 = \beta^{-2} \frac{\partial^2}{\partial H^2} \ln Z_N = k_B T \chi_T.$$

Solution: