## [tex160] MB gas with ultrarelativistic energy-momentum relation

Consider a very hot and dilute gas with uniform density and pick out a region of volume $V$. The ultrarelativistic energy-momentum relation $\epsilon(\mathbf{k}) = c\hbar|\mathbf{k}|$ is assumed to be valid for all particles. The Maxwell-Boltzmann grand potential can then be calculated as the following sum over one-particle energy levels:

$$\Omega = -k_{\rm B}Tz\sum_{\mathbf{k}} e^{-\beta\epsilon(\mathbf{k})}, \tag{1}$$

where $z = e^{\mu/k_{\rm B}T}$ is the fugacity and $\mu$ the chemical potential. The wave vectors $\mathbf{k}$ are uniformly distributed in reciprocal space with density $\rho(\mathbf{k}) = V/(2\pi)^3$.
(a) Show that the density of energy levels is

$$D(\epsilon) = \frac{4\pi V}{(hc)^3}\epsilon^2. \tag{2}$$

(b) Convert the sum in (1) into an integral using (2) and infer from that integral the following expression for the grand potential:

$$\Omega(T,V,\mu) = -\frac{8\pi V}{(hc)^3}z(k_{\rm B}T)^4. \tag{3}$$

(c) Infer from (3) expressions for pressure $p(T,\mu)$, entropy $S(T,V,\mu)$, and average number of particles $N(T,V,\mu)$ via partial derivatives. Show from the resulting expressions that the equation of state $pV = Nk_{\rm B}T$ holds.
(d) Construct the expression for the internal energy via $U = \Omega + TS + \mu N$ and show that it is consistent with $U = 3Nk_{\rm B}T$.
(e) Calculate the Helmholtz free energy $A(T,V,N) = U - TS$ and confirm that the result is consistent with the one obtained in [tex77] in the framework of a canonical ensemble.
Take note of the integral $\int_0^\infty dx\, x^n e^{-x} = n!$.

**Solution:**