[tex194] Ising lattice gas in $\mathcal{D} = 1$: equation of state

We consider a linear array of N cells of volume v_c. Each cell contains at most one particle. Particles in nearest-neighbor cells experience an attractive force, producing an interaction energy $-u$. The grand partition function $\mathcal{Z}(T, V, \mu)$ of the lattice gas is related to the canonical partition function $Z_N(T, H)$ of an Ising chain of N sites as established in [tsc18]:

$$\mathcal{Z}(T, V, \mu) = e^{-N\beta(H+J)} Z_N(T, H) = e^{-N\beta H} \left[\cosh(\beta H) + \sqrt{\sinh^2(\beta H) + e^{-4\beta J}}\right]^N, \qquad (1)$$

where $J = u/4$, $H = -(u + \mu)/2$, $\beta = 1/k_B T$, and $N = V/v_c$.
(a) Derive from $\mathcal{Z}$ the parametric expression,

$$\frac{pV}{Nk_BT} = w + \ln\left(\cosh w + \sqrt{\sinh^2 w + e^{-\beta u}}\right), \quad \frac{N_\mathrm{p}}{N} = \frac{1}{2}\left[1 + \frac{\sinh w}{\sqrt{\sinh^2 w + e^{-\beta u}}}\right], \qquad (2)$$

for the thermodynamic equation of state, where $w = \beta(u + \mu)/2$ is the parameter.
(b) Show that in the limit $u \to 0$ we obtain the explicit result,

$$\frac{pV}{N_\mathrm{p} k_B T} = -\frac{N}{N_\mathrm{p}} \ln\left(1 - \frac{N_\mathrm{p}}{N}\right), \qquad (3)$$

for the ideal lattice gas (valid for any configuration of cells).

Solution: