[tex57] Energy distribution for N ideal gas atoms.

The equilibrium velocity distribution for N atoms of a classical ideal gas is

$$f(\mathbf{v}_1, \ldots \mathbf{v}_N) = \left(\frac{m}{2\pi k_B T}\right)^{3N/2} e^{-m(v_1^2+\cdots+v_N^2)/2k_B T},$$

where $\mathbf{v}_i = (v_{ix}, v_{iy}, v_{iz})$.
(a) Determine the associated energy distribution $f_E(E)$, where $E = \frac{1}{2}m(v_1^2 + \cdots + v_N^2)$.
(b) Define the function $F_n(x)$ via $F_n(x)dx = f_E(E)dE$ with $x = E/nk_B T, n = 3N/2 - 1$. Show that this function is

$$F_n(x) = \frac{n^{n+1}}{\Gamma(n+1)} x^n e^{-nx}.$$

(c) Use Stirling's approximation to show that for large n we have

$$F_n(x) \rightsquigarrow F_n(x)_{\rm as} = \sqrt{\frac{n}{2\pi}} \left[x e^{1-x}\right]^n.$$

(d) Plot $F_n(x)$ and $F_n(x)_{\rm as}$ in the same graph for $0 < x < 4$ and $N = 1, 2, 5, 10, 20$ and comment on the deviations. Plot $F_n(x)_{\rm as}$ separately for $0 < x < 4$ and $N = 100, 250, 500, 1000$ and comment on the trend as n becomes large.

Solution: