[tex91] Relativistic ideal gas (canonical partition function)

Consider a classical ideal gas of N atoms confined to a box of volume V in thermal equilibrium with a heat reservoir at a very high temperature T. The Hamiltonian of the system,

$$H = \sum_{l=1}^{N} \left[\sqrt{m^2c^4 + p_l^2c^2} - mc^2 \right],$$

reflects the relativistic kinetic energy of N noninteracting particles. Here c is the speed of light and $p_l = |\mathbf{p}_l|$ is the magnitude of the momentum of particle l.
(a) Show that the canonical partition function can be expressed in the form

$$Z_N = \frac{1}{N!} \left[4\pi V \left(\frac{mc}{h} \right)^3 \frac{e^u}{u} K_2(u) \right]^N, \quad u \equiv \beta mc^2, \quad K_\gamma(u) = \frac{u}{\gamma} \int_0^\infty dx\, \sinh x \sinh(\gamma x) e^{-u \cosh x}$$

where $K_\gamma(u)$ is a modified Bessel function.
(b) Recover the result from [tex76] for Z_N of the nonrelativistic ideal gas at $k_BT \ll mc^2$ by using the asymptotic expression $K_2(u) \simeq \sqrt{\pi/2u}e^{-u}$ for $u \gg 1$.
(c) Recover the result from [tex77] for Z_N of the ultrarelativistic ideal gas at $k_BT \gg mc^2$ by using the asymptotic expression $K_2(u) \simeq 2/u^2$ for $u \ll 1$.

Solution: